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SOME NEW METRICAL PROPERTIES OF CONIC CURVES.

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Theorem 1. If through the extremities of a double ordinate to any diameter of an ellipse right lines be drawn from the extremities of said diameter; then such lines will ever intersect upon a hyperbola, having said diameter and its conjugate in common with the ellipse; and thus asymptotes which are both parallel to and bisected by the supplementary chords uniting said conjugate diameters. While lastly, the double ordinate of the said point on the hyperbola, parallel to the given double ordinate in the ellipse, will cut, with it, their common bisecting diameter harmonically.

And conversely, of course, lines through the extremities of any double ordinate in a hyperbola from the ends of its diameter, will meet upon an ellipse having said diameter and its conjugate in common; and the ordinates of the corresponding points will cut their common diameter harmonically.

For let pTq be a double ordinate in an ellipse [Fig. 1], cutting its bisecting diameter DCD' in T; and let $D'pP$ and qDP meet in P. Draw PNQ parallel to pTq, meeting DCD' in N, and pD in Q. And draw ECE' the conjugate diameter to DCD'.

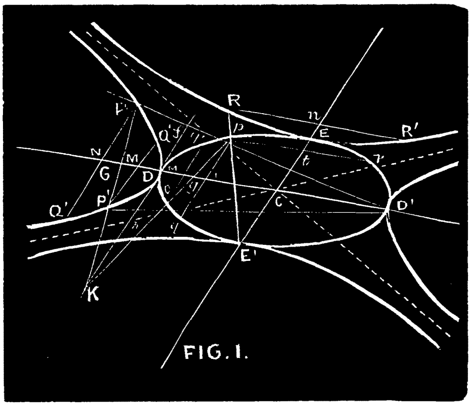

FIG. I.

Then qTD and PND being similar triangles; and likewise pTD' and PND'; while $qT=pT$, and $PN=QN$; therefore $pT:DT=qT:DT=PN:DN$, while $pT:D'T=PN:D'N$. So that $pT^2:DT.D'T=PN^2:DN.D'N=QN^2:DN.D'N=CE^2=CD^2$; which latter ratio belongs to both ellipse and hyperbola. And thus P and Q must ever lie on a hyperbola, having DCD' and ECE' for its conjugate diameters also.

In a precisely similar way, were ptr, parallel to DCD', the double ordinate taken; and lines $E'pR$ and rER drawn through p and r from the ends of its bisecting diameter ECE'; then such lines $E'pR$ and rER can be shown to meet upon a hyperbola, having ECE' and DCD' as its conjugate diameters also; and which is thus the curve conjugate to the first hyperbola.

And conversely, of course; were the double ordinate PNQ, or RnR' taken in the hyperbola, lines PpD' and QDp, or RpE' and $R'Ep$, can be shown to meet upon point p on an ellipse, having DCD' and ECE' for its conjugate diameters.

And next; the conjugate diameters DCD' and ECE' being common to both ellipse and hyperbola, by a law of the latter curve, the supplementary chords ED' and $E'D'$ must be parallel to one asymptote, and be bisected by the other.

While lastly, the parallel double ordinates PNQ and pTq [or RnR' and ptr] divide their common bisecting diameter DCD' [or ECE'] harmonically. For $DT:DN=qT:PN=pT:PN=D'T:D'N$. And similarly $Et:En=E't:E'n$.

Scholium. But note. In all the foregoing we have considered, in the hyperbola, merely those double ordinates that cut *one* branch. Those which cut *both* branches will be dealt with later, in Theorem 5. From which it will be evident that the true analogue, in the hyperbola, to the minor ordinates of the ellipse, is not its own minor ordinate, cutting both branches; but is rather the major ordinate of its conjugate.

Corollary 1. The four tangents at $DD'E$ and E' are common to ellipse and hyperbola.

Corollary 2. PT and QT will be the tangents to the hyperbola from T; and Rt, $R't$ the tangents from t to its conjugate. While in like manner pN, qN will be the tangents to the ellipse from N; and pn, rn its tangents from n.

Corollary 3. The parallel double ordinates pTq and PNQ, or ptr and RnR', through any two such corresponding points pP, or pR, must therefore cut the curves again in two fresh corresponding points qQ, or rR'. So that the determining lines to such double ordinates can be drawn through either extremity of its diameter indifferently. As *e. g.* $D'pP$ and $D'qQ$; or pDQ and qDP; $E'pR$ and $E'rR'$; or rER and pER'.

Corollary 4. When the axi of an ellipse are taken as our conjugate diameters, then the resultant hyperbola will have its axi either coincident, or reversed, according as the double ordinate in the ellipse was major or minor.

[NOTE. As first drafted, the foregoing and following theorems and corollaries considered merely this special case. But acting upon a hint from Professor Dickson, that by Projectional Geometry, Theorem 1 was true for *any* diameter, all were readily thrown into their present general form, by merely reading conjugate diameters for axi.]

Corallary 5. When the common conjugate diameters are equal, then evidently the projected hyperbola is equilateral, with the produced axi of the ellipse for its asymptotes; and conversely.

Corollary 6. While if a circle be taken as our ellipse, the hyperbola is again an equilateral.

Corollary 7. But on the other hand, when the ellipse passes into a parabola, then the projected conic has become a second and equal parabola, reversed in direction.

For obviously, if pTq be a double ordinate of a parabola [Fig. 2] and CTD its bisecting diameter, then since $pT=qT$, $pP=2DT=qQ=TN$; so that $pPQq$ is a parallelogram, for any position of the double ordinate pTq. Hence PQ lie on a parabola, equal to pDq, with $CTDNC'$ as a common diameter; and fDh as a common tangent. While DT ever equals DN; and pN, qN are the tangents from N to pDq; even as PT, QT are those from T to PDQ. While PQ lie on, and are definable by lines pDQ and qDP; as well as on lines pP and qQ, parallel to CDC'.

Theorem 2. If then pP and $p'P'$, or pP and $p'Q'$, be two pairs of corresponding points, either upon an ellipse and its projected hyperbola [Fig. 1], or upon a pair of equal parabolas [Fig. 2], said points being determined, either by the double ordinates pTq and PNQ, $q'mp'$ and $Q'MP'$, through the harmonic points TN and mM upon diameter DCD'; or else given by the intersection of the two curves by the lines $D'pP$ and $D'p'P'$ [or pDQ and $p'DQ$] through a common extremity D' [or D] of said diameter DCD'. Then the resultant corresponding chords pp' and PP' [or pp' and QQ'] both cut the diameter in two fresh harmonic points g and G; and also meet upon the tangent DK [or $D'K'$] of that extremity of the diameter through which the determining lines were not drawn.

For let chords pp' and PP' be taken in the ellipse and hyperbola [Fig. 1] determined by lines $D'pP$ and $D'p'P'$. Draw the double ordinates pTq, $p'mq'$, $P'MQ'$ and PNQ; and let the tangents at D cut $D'pfP$ in f; and $D'p'h'P'$ in h. Then T and N must be harmonic points [Theorem 1, Corollary 3]; and likewise m and M. Therefore $fp:fP=DT:DN=D'T:D'N=D'p:D'P$; and $hp':hP'=Dm:DM=D'm:D'M=D'p':D'P'$. So that $D'pfP$ and $D'p'hP'$ are harmonic ranges, whose pencil rays pp', fh, PP' must concur in K a summit of their quadrilateral; the other two summits being D', and a second point upon the tangent $fDhK$.

And in like manner, were chords pp' and QQ' chosen, they can be shown to concur upon the tangent at D'; the other two summits of their quadrilateral being D, and a second point upon the tangent at D'. So that in both cases $Dg:D'g=DG:D'G$.

While if chords pq' and PQ' [or QP'], determined by lines $D'pP$ and $D'q'Q'$ [or pDQ and $q'DP'$] be the ones taken; these can again be shown to meet upon the tangent at D [or D']; and to cut harmonically DCD' in G and g. And all the foregoing is, of course, equally true for any corresponding chords in the ellipse and conjugate hyperbola, that cut diameter ECE'.

Lastly; in the pair of equal but opposite parabolas [Fig. 2], which have diameter CDC' and its tangent fDh in common, let the chords pp' and PP' be taken; determined by lines pP and $p'P'$ parallel to CDC'. Draw the double ordinates pTq, $p'mq'$, $P'MQ'$, and PNQ; let the tangent of D meet pfP in f and $p'hP'$ in h; and let pp' and PP' cut diameter CDC' in g and G, respectively. Then since $pf=fP$, and $p'h=h'P$, pp' and PP' must concur in K on $fDhK$; while $gD=DG$.

Were pp' and QQ' the chosen chords [Fig. 2], determined by lines pDQ and $p'DQ'$; then since $pD=DQ$ and $p'D=DQ'$; pp' and QQ' must be parallel; and again $gD=DG$. While if we chose chords pq' and PQ', they must again concur on the tangent of D; and cut diameter CDC' at equal distances from D.

Corollary 1. If, therefore, any double ordinate in the ellipse, say pr, be considered with relation to its parallel diameter [*e. g.* DCD']; since they cut at infinity, the corresponding chord in the hyperbola, determined by lines from an extremity of the same diameter [*i. e.* $D'p$ and $D'r$; or Dp and Dr], must pass through the common center of the curves. And conversely, of course, if a hyperbola's double ordinate is considered in relation to its parallel diameter, then its corresponding chord in the ellipse passes through the common center C.

Theorem 3. But if in place of both determining lines running through the same extremity of a diameter, we have one through each extremity—as *e. g.* Fig. 3, $D'pP$ and DqQ,—then the connecting chords pq and PQ will cut the diameter in the *same* point; which will be internal or external, according as the two pairs of chosen points fell upon the opposite, or the same side of said diameter.

For, let pq and PQ be the chosen chords [Fig. 3], in ellipse and hyperbola, connecting the pairs pP and qQ determined by $D'pP$ and DqQ. And let the tangent at D cut $D'pP$ in f; while that at D' cuts DqQ in h. Draw PN, pT, qt, Qn, the ordinates of points Ppq and Q to diameter $D'CD$.

Then, as before, $fp{:}fP=DT{:}DN=D'T{:}D'N=D'p{:}D'P$; while $hq{:}hQ=D't{:}D'n=Dt{:}Dn=Dq{:}DQ$. So that $D'pfP$, and $DqhQ$ are harmonic ranges; and thus, if DD', pq, fh, and PQ be joined, they must concur in G upon DD'.

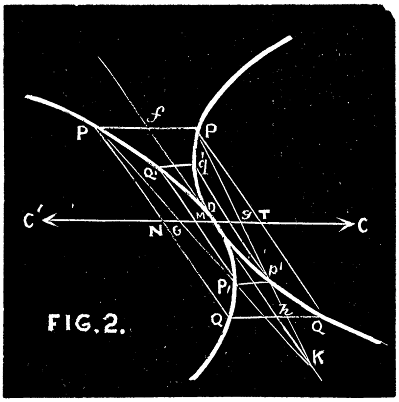

FIG. 2.

And similarly; if $p'P'$ and qQ be chosen on the same side of DCD', their chords $p'q$ and $P'Q$, in like manner, can be shown to concur in H on DCD' produced. Or if points on the ellipse and the conjugate hyperbola be chosen, determined by lines through E and E', their chords will concur on diameter ECE'.

While in the pair of parabolas [Fig. 2], if chords pq' and PP' be chosen, determined by lines pfP and $q'DP'$; then again, since $pf=fP$; and $q'D=DP'$, the pencil rays pq' and PP' must concur in G upon CDC'; and fG be parallel to $q'DP'$, since they meet upon the ideal tangent at D' at infinity. Likewise PQ' and pp' concur in g; and fg is parallel to $Q'Dp'$.

Theorem 4. And hence, since the tangents of, or chords through corresponding points upon an ellipse and its auxiliary circle, determined by their common major ordinates, concur on the major axis. And similarly, those through the corresponding points upon the ellipse, and a circle whose diameter is the minor axis, determined by their common minor ordinates, concur on the minor axis.

So in like manner, if we draw the tangents of, or chords through the corresponding points upon a hyperbola and its auxiliary circle; or upon the conjugate and its auxiliary circle; said corresponding points being determined by the polar ordinates to the respective major axis; then the said corresponding tangents or chords will cut that axis harmonically [Theorem 2].

Except where the chord of the hyperbola cuts both branches, and each extremity is also upon the same side of the respective major axis, as the corresponding extremity of the circle's chord. In which special case the said corresponding chords will cut the axis in the *same* point [Theorem 3].

[*Scholium.* And note here again, that the hyperbola which bears the same relation to the circle on the minor axis as the ellipse does to its minor auxiliary circle, is the *conjugate* hyperbola.]

Theorem 5. If from the extremities of a double ordinate in a hyperbola, which cuts both branches of the curve, right lines be drawn through the extremities of its bisecting diameter, these will intersect upon a second point on the curve; the double ordinate through which will cut said diameter harmonically to the first.

Let QyQ' be a double ordinate [Fig. 3], cutting both branches; with ECE' as its bisecting diameter; and let QRE and $Q'E'R$ intersect in R. Draw RvP' parallel to QyQ', and meeting ECE' in v.

Then by similar triangles, $Qy:yE' = Q'y:yE' = Rv:vE'$; and $Qy:yE = Rv:vE$. So that both $yE':yE = vE':vE$, and also $Qy^2:yE'.yE = Rv^2:vE'.vE$; or $Qy^2:Rv^2 = Cy^2 - CE'^2 : CE'^2 - Cv^2$ [Euclid II., 5 and 6].

But since yv divide EE' harmonically, $Cy:CE' = CE':Cv$. For $yE':yE = vE':vE$; or $Cy - CE' : Cy + CE' = CE' - Cv : CE' + Cv$; so that $Cy:CE' = CE':Cv$ [Euclid V., E]. And hence $Cy^2:CE'^2 = CE'^2:Cv^2 = Cy^2 + CE'^2 : CE'^2 + Cv^2 = Cy^2 - CE'^2 : CE'^2 - Cv^2$. And thus $Qy^2:Rv^2 = Cy^2 + CE'^2 : CE'^2 + Cv^2$; or $Qy^2 : Cy^2 + CE'^2 = Rv^2 : CE'^2 + Cv^2$.

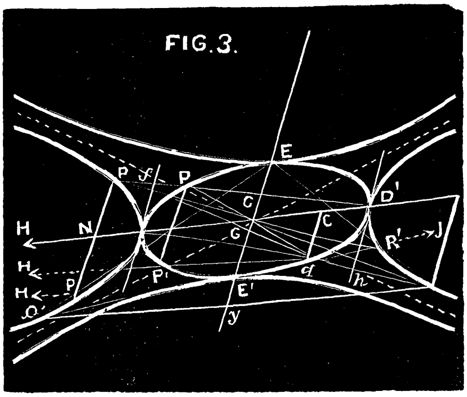

FIG. 3.

Now by a well known theorem, $D'n.Dn:Qn^2 = CD'^2:CE'^2 = Cn^2 - CD'^2 : Qn^2 = Qy^2 - CD'^2 : Cy^2 = Qy^2 : Cy^2 + CE'^2$; which as we have seen $= Rv^2 : CE'^2 + Cv^2$. And thus Rv is also an ordinate to diameter ECE'; and hence R is a point upon the curve.

And conversely, of course, if RvP' were the double ordinate chosen, cutting diameter ECE' internally, then ERQ and $P'E'Q$ can be shown to meet in Q upon the curve, whose ordinate Qy will cut ECE' externally, and harmonically to v.

Corollary 1. If the tangent at E' cuts QRE in say w; then $Qw:QE = yE':yE = vE':vE = Rw:RE$. So that $QwRE$ will ever be a harmonic range.

Corollary 2. And hence when Q is taken at infinity, and thus QE becomes parallel to its asymptote; R must coincide with D', and $D'w=D'E$.

Corollary 3. While on the other hand, when Q coincides with the intersection of the curve by the tangent at E', then QRE must cut the curve infinitesimally close to Q; or in other words, has become the tangent at Q. A result also directly evident from the general law that if Pt be the tangent, and Pn the normal to any diameter ECE'; then $Ct.Cn=CE^2$.

Corollary 4. Therefore E [or E'] is a pole, and the tangent at E' [or E] its polar, for the hyperbola $QRD'PDQ'$. And similarly, D [or D'] is a pole, and the tangent at D' [or D] its polar, for the conjugate curve.

Corollary 5. And thus if any line through E [or E'] cut the curve in RQ, then $E'R$ and $E'Q$ [or ER and EQ] will cut the conjugate diameter DCD' in say JH at equal distances from C. For $RE'Q'$ being collinear, $CH:CJ=Qy:Q'y$. And thus if EJ be joined, CEJ and $CE'H$ will be equal triangles, and EJ be parallel to $QE'H$; and similarly, EH parallel to $E'RJ$. So that if L be the point where ERQ cuts DCD', then ELJ and QLH are similar triangles; and $EL:LQ=JL:LH$.

Theorem 6. If pp' be any chord in a conic curve, DCD' its bisecting diameter, and q any other point on the curve; then chords pq and $p'q$ will ever cut DCD' harmonically. Except when the chosen chord pp' is one in a hyperbola, which cuts both branches. In which case one of the said "harmonic segments" of the bisecting diameter will be reversed in direction.

Let pp' be a chord in the ellipse [Fig. 3], DCD' its bisecting diameter, and q any other point on the curve. Let pq and $p'q$ cut DCD' in G and H, respectively. Let PP' be the chord and Q the point in the projected hyperbola, corresponding to pp' and q; and defined by lines pDP', $p'DP$, and DqQ. Then chord PP' must be also definable by lines $D'pP$ and $D'p'P'$ [Theorem 1, Corollary 3]. And thus while chords pq and $P'Q$ cut DCD' harmonically in G and H [Theorem 2]; chords pq and PQ, on the other hand, must cut it in the same point G [Theorem 3]; while $p'q$ and $P'Q$ cut it in H. So that chords pq and $p'q$ in the ellipse cut DCD' harmonically; and likewise chords PQ and $P'Q$ in the hyperbola.

In the same way the lines joining any point to the extremities of any chord cutting but one branch in the conjugate hyperbola cut the bisecting diameter harmonically. While in the twin parabolas [Fig. 2], if pq be the chord, bisected by CDC', and p' be the point; pp' and $p'q$ will cut CDC' harmonically. For let PQ be the chord and P' the point in the second parabola, defined by lines pP, $p'P'$, and qQ, parallel to CDC'. Then PQ also lie upon lines qDP and pDQ [Theorem 1, Corollary 7]. So that chords qp' and QP' will cut CDC' harmonically in Gg [Theorem 2]; while pp' and QP' will cut it in the same point g [Theorem 3], and thus pp' and qp' cut CDC' harmonically.

Lastly; let the chosen chord in the hyperbola be one QyQ' cutting both branches of the curve [Fig. 3], with ECE' as its bisecting diameter, and P as the point. Then chords PQ and PQ' will cut ECE' in say z and z', respectively; coincident with its intersection by the ordinate and tangent of some fourth point on the curve. So that $Cz.Cz'=CE^2$; and if either Cz or Cz' be reversed in direction, ECE' will be cut harmonically.

For if R or P' were taken as our fixed point; then as shown in Theorem 5, RQ and RQ', or $P'Q'$ and $P'Q$ will cut ECE' in E and E'. While if a point on the curve—say Q''—cut by the chord through Q or Q', parallel to ECE', were chosen; then obviously, chord $Q''Q'$, or $Q''Q$ must pass through C. While lastly, if the fixed point be taken infinitesimally close to either Q or Q', then its lines through Q and Q' have become the tangent and double ordinate of said points; and thus the product of their segments again equals CE^2.

But since this is now true for *six* points on the curve, by the principle of continuity it must be true for *any* point on said curve. And hence PQ and PQ' must cut ECE' in say z and z'; so that $Cz.Cz'=CE^2$; the tangent to the ordinate of z to ECE' passing through z'; and conversely; and thus if either Cz or Cz' be reversed in direction, ECE' will be cut harmonically.

Corollary 1. Therefore in the ellipse chords pB and pB', through the extremities of the minor axis BB', will ever cut AA', the major axis, harmonically. And similarly, pA, pA' cut BB' harmonically.

Corollary 2. While in the hyperbola, since lines through P parallel to the asymptotes will represent both the lines to the *ideal* minor axis, and those to the *ideal* diameter conjugate to DCD', both at infinity. Therefore such lines will cut both DCD' and the major axis AA' to the curve harmonically; and will intersect both ECE', and the accepted minor axis BB'—the major of the conjugate—in the same distances from C, though one of them reversed, as are a harmonic pole and polar.

And in like manner PA, PA' will ever cut BB' in its intersection by the ordinate and tangent of the same fourth point on the curve; even as PD, PD' also cut ECE'.